A condition for absence of eigenvalues

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Abstract

In this paper we show that Schrödinger operators with a certain type of potentials has no eigenvalues when the mass μ is sufficiently small.

1 Introduction

In this paper we are going to treat with Schrödinger operators $H(\mu)$ as follows.

$$H(\mu) = -\frac{\Delta}{2\mu} + V, \tag{1}$$

where μ is the mass and V is the potential.

There is a great amount of studies on the existence of eigenvalues of Schrödinger operators [2]. For example, it is well known that there exists no eigenvalues when the potential is repulsive ([1], p. 236).

In this paper we present a simple proof of the nonexistence of eigenvalues of Schrödinger operators with a certain type of potentials when the mass μ is sufficiently small. We will exploit the equation which we derived in [3], [4].

2 Results

First we assume that $H(\mu)$ has an eigenvalue E, that is

$$-\frac{\Delta}{2\mu}\Psi + V\Psi = E\Psi \tag{2}$$

where

$$\int \Psi(x)\Psi(x)\,dx = 1. \tag{3}$$

V is a multiplication operator of a function $V(x)$ which satisfies the following assumption.

(A1) Let $V(x)$ be a $C^\infty(R^3/\{0\})$ function.

Here we present our previous result ([3], [4]).

Theorem 1

In addition to assumption (A1), let the potential $V(x)$ satisfy the following condition.

(A2) There exist constants $M \geq 0$ and $N > 0$ such that

$$|G(x)| \leq M|V(x)| + N \tag{4}$$

where

$$G(x) \equiv \sum_{l=1}^{3} x_l \frac{\partial V}{\partial x_l} \tag{5}$$

Then for any real number α

$$\frac{dE(\mu)}{d\mu} = -\frac{2(2\alpha-1)}{(2\mu)^2}(\Delta\Psi,\Psi) - \frac{\alpha}{\mu}(G\Psi,\Psi) \tag{6}$$

If one puts $\alpha = 0$ in (6), then

$$-\frac{dE(\mu)}{d\mu} = 2(2\mu)^{-2}(-\Delta\Psi,\Psi) = 2(2\mu)^{-2}\sum_{j=1}^{3}\Big(\frac{\partial\Psi}{\partial x_j},\frac{\partial\Psi}{\partial x_j}\Big) \tag{7}$$

Therefore $\frac{dE}{d\mu}$ is always negative.

Now we are ready to state the result.

Theorem 2
In addition to the assumptions of theorem 1, let the potential $V(x)$ satisfy the following conditions.

(A3) There exist positive numbers θ, C_1 such that

$$G(x) + \theta V(x) \le \frac{C_1}{|x|^2}$$
$$0 < \theta < 2$$

(A4) There exists a positive number C_2 such that

$$-\frac{C_2}{|x|^2} \le V(x)$$

Then there exists a positive number μ_0 such that $H(\mu)$ has no eigenvalues for positive μ smaller than μ_0.

Proof
Insertion of (2) into (6) yields the following equation.

$$\mu\frac{dE}{d\mu} + (1-2\alpha)E = (((1-2\alpha)V - \alpha G)\Psi,\Psi) \tag{8}$$

Putting $\alpha = 1/(2-\theta)$, we get

$$\mu\frac{dE}{d\mu} - \frac{\theta}{2-\theta}E = -\frac{1}{2-\theta}((G+\theta V)\Psi,\Psi) \tag{9}$$

Using assumption (A3), we get

$$\begin{aligned}\mu\frac{dE}{d\mu}-\frac{\theta}{2-\theta}E &\geq -\frac{1}{2-\theta}\Big(\frac{C_1}{|x|^2}\Psi,\Psi\Big)\\ &\geq -\frac{4C_1}{2-\theta}(-\Delta\Psi,\Psi)\\ &= \frac{8C_1}{2-\theta}\mu^2\frac{dE}{d\mu}\end{aligned}$$

Hence

$$\Big(1-\frac{8C_1\mu}{2-\theta}\Big)\mu\frac{dE}{d\mu}-\frac{\theta}{2-\theta}E\geq 0 \tag{10}$$

Here we have used the following well-known inequality

$$\int_{R^3}\frac{|u(x)|^2}{|x|^2}\,dx\leq 4\sum_{j=1}^{3}\int_{R^3}\left|\frac{\partial u}{\partial x_j}\right|^2\,dx \tag{11}$$

and (7).

Since $\frac{dE}{d\mu}$ is negative, it follows from (10) that $E(\mu)$ is negative for sufficiently small μ.

Next we are going to present the opposite assertion.

Using assumption (A4), we get

$$(V\Psi,\Psi)\geq\Big(-\frac{C_2}{|x|^2}\Psi,\Psi\Big)\geq 4C_2(\Delta\Psi,\Psi)$$

Here we have again used inequality (11).

It follows from (2) that

$$\begin{aligned}E &= (V\Psi,\Psi)-\frac{1}{2\mu}(\Delta\Psi,\Psi)\\ &\geq\Big(-4C_2+\frac{1}{2\mu}\Big)(-\Delta\Psi,\Psi)\end{aligned}$$

Therefore $E(\mu)$ is positive for sufficiently small μ.

Consequently we have a contradiction, which means the absence of eigenvalues for sufficiently small μ.

Remark

Here we show an example of the potentials which satisfy the assumptions of theorem 2. That is,

$$V(x)=-b\frac{e^{-a|x|}}{|x|^d}$$

where

$$0<d<2,\quad a>0,\quad b>0.$$

By elementary calculation we get

$$G(x) = b\frac{(a|x| + d)e^{-a|x|}}{|x|^d}$$

Therefore it is obvious that $V(x)$ satisfies the assumptions.

References

[1] **P. D. Hislop, I. M. Sigal**: Applied Mathematical Sciences, <u>113</u> Introduction to Spectral Theory(1995)

[2] **M. Reed and B. Simon**: Methods of Modern Mathematical Physics <u>4</u>

[3] **H. Uematsu**: Bulletin of the Association of Natural Science Senshu University <u>33</u> (2002) p. 21

[4] **H. Uematsu**: Bulletin of the Association of Natural Science Senshu University <u>34</u> (2003) p. 1